Canonical perturbative approach to nonlinear systems with application to optical waves in layered Kerr media

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(Received 29 December 1998; revised manuscript received 7 September 1999)

We investigate electromagnetic wave reflection and propagation in layered Kerr structures by introducing a method based on the application of canonical perturbation theory to fields in nonlinear media. Via the Hamilton-Jacobi formalism of classical mechanics, the waves in linear layers are expressed with constant canonical variables. The nonlinearity is treated as a small perturbation that modifies the constant invariants. We explicitly evaluate the nonlinear fields correct to first order by perturbation and compare the results to a rigorous nonlinear thin-layer model. Both polarizations, TE and TM, are considered separately. An exact quadrature solution of the nonlinear field in TM polarization is derived. We show that with weak nonlinearities the perturbative technique yields simple and accurate analytical expressions for the nonlinear fields. The results give physical insight into the use of nonlinear media for controlling the scattered fields in layered structures.

PACS number(s): 42.65.Tg, 46.05.+b, 78.20.Bh

I. INTRODUCTION

Nonlinear equations of evolution are frequently encountered in optics [1,2]. They occur as a consequence of the interaction of intense laser light with matter. An especially important nonlinear phenomenon is the Kerr effect in which the refractive index of the medium depends linearly on the electric-field intensity. Optical bistability [3] and particlelike solutions, solitons [4], have many potential applications in optical communication [5] and optoelectronic devices [6]. Depending on the polarization of the incident wave the Kerr-Maxwell equations reduce to two different nonlinear equations; in TE polarization they take on a relatively simple form of a nonlinear Helmholtz equation, while in TM polarization the nonlinearity couples to the fields in a considerably more nontrivial manner. In some specific cases exact wave solutions can be found, but usually most theoretical considerations are based on approximate techniques. If the nonlinearity is sufficiently weak the results are expected to be nearly exact.

The electromagnetic field propagation is conventionally analyzed in layered and stratified structures. The fabrication of such multilayered elements is technically possible, and for instance, optical bistability by excitation of a nonlinear guided wave is readily observed in experiments [7]. From the theoretical point of view layered media simplify the field equations and some nonlinear systems become analytically soluble [8]. The (non)linear electromagnetic fields are usually represented by plane waves [9]. This involves an approximation that the field-envelope variation occurs slowly, over distances much larger than an optical wavelength [10]. However, in many situations the nonlinear equations are solved using perturbative techniques. There exist two main practices: either these equations are linearized by expanding the solutions about the unperturbed ones [11], or the perturbation method is based on the inverse scattering theory [12].

We take a new approach to the perturbative nonlinear theories. We investigate layered structures that include nonlinear Kerr media. For simplicity, we consider a system of three dielectric layers in which the middle layer is taken to be a Kerr medium. This geometry is common in optoelectronic systems [13] and it describes, for example, a nonlinear mirror or waveguide. We calculate the nonlinear electric and magnetic fields perturbatively using the canonical Hamilton's theory [14,15]. To our knowledge this technique is an altogether new application of classical mechanics in optics. From the Hamilton-Jacobi equation [14] we find the exact linear electromagnetic solution in terms of canonical variables that are constants of integration. The nonlinearity is treated as a small perturbation that modifies the constant invariants. With weak nonlinearities this assumption is physically justified. We establish the first-order corrections by explicit calculations and assess the accuracy of the results using the rigorous nonlinear thin-layer theory [16]. Both TE and TM polarizations are considered separately. An exact quadrature solution of the TM-polarized nonlinear field is also derived. We show that the perturbative approach leads to accurate analytical solutions for the nonlinear fields. Our results give analytical insight into the nonlinear wave behavior and demonstrate, for example, that in layered structures the nonlinearity can be used to manipulate the scattered fields.

This paper is organized as follows. In Sec. II we introduce the model and the main nonlinear equations for both polarizations. In Sec. III we apply Hamilton's canonical perturbation theory to the fields in a Kerr medium and find the first-order solutions. In Sec. IV we briefly describe the nonlinear thin-layer theory. The numerical results are presented and discussed in Sec. V. Finally, in Sec. VI we summarize the main conclusions.

II. NONLINEAR MODEL

Layered geometries are elementary structures that are frequently employed in optical systems. The applications in-

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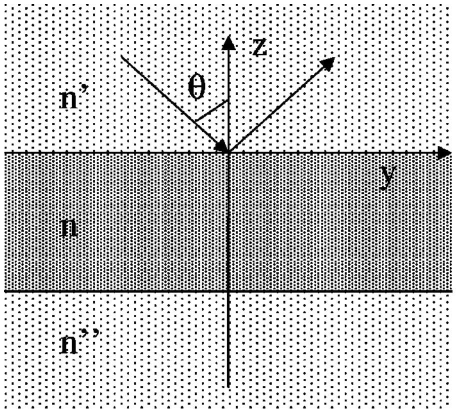

FIG. 1. Layered geometry of the nonlinear model. The angle of incidence is θ, $n'=n''$ and n are the refractive indices of the linear and nonlinear media, respectively.

clude planar waveguides, optical switches, sensors, etc. When the medium contains nonlinearities, the investigations of the rigorous electromagnetic solutions become effectively more complex. The exact expressions [8] usually also are too complicated to be of value in practical applications.

We take an alternative approach to the exact nonlinear theories. We aim to find simple but accurate analytical forms for the nonlinear fields. We first introduce the model employed in this paper and present the nonlinear field equations whose perturbative solutions within the Hamiltonian formalism are examined in Sec. III. Both polarizations, TE and TM, are considered separately.

A. Geometry

The nonlinear system is illustrated in Fig. 1. For simplicity, we have taken it to consist of three dielectric layers only. The indices of refraction are denoted by n', n, and n''. The middle layer is assumed to be a Kerr medium, while the other two layers are linear and $n''=n'$. An incident wave of frequency ω propagates at an angle θ to the normal of the structure. If the middle layer is linear and its thickness is properly chosen, one may obtain a situation in which all light traverses the structure and the reflected intensity is zero.

We study the effects of the nonlinearity to the scattered fields. This is an interesting and important application of layered structures. Within the middle layer the waves satisfy two different nonlinear equations depending on the state of polarization of the incident light. In our approach, we solve these nonlinear propagation equations perturbatively using Hamilton's canonical theory (Sec. III).

B. TE polarization

For a TE-polarized (s) wave the electric field is perpendicular to the plane of incidence. By starting from Maxwell's equations (in Gaussian units with unit magnetic permeability), one may derive a nonlinear equation for the electric field $E=E(y,z)$,

$$\frac{\partial^2 E}{\partial y^2}+\frac{\partial^2 E}{\partial z^2}+n^2 E=0, \tag{1}$$

where the effective index of refraction n satisfies the Kerr law,

$$n(y,z)=n_0+n_2|E(y,z)|^2. \tag{2}$$

Here $|E(y,z)|^2$ is the optical intensity $|E|^2=EE^*$, n_0 is the refractive index of the linear medium, and n_2 is a small real coefficient that describes the strength of the nonlinearity. For simplicity, we use scaled forms of the coordinates, i.e., $y\rightarrow y/k_0$ and $z\rightarrow z/k_0$, where $k_0=\omega/c$ and c is the speed of light in vacuum.

Especially, we want to study systems for which the fields vary as plane waves along the y coordinate, i.e., $E(y,z)=E(z)e^{ik_y y}$, where $k_y=n'\sin\theta$ is the wave vector (in units of k_0) in y direction. The refractive index n then becomes a function of z only, $n(y,z)=n(z)$, and Eq. (1) assumes the form

$$\frac{\partial^2 E}{\partial z^2}+c_1 E+c_2|E|^2 E=0, \tag{3}$$

where $c_1=n_0^2-k_y^2$ and $c_2=2n_0n_2$. We call Eq. (3) the TE-polarized Kerr-Maxwell equation, and it is one of the nonlinear equations that we solve perturbatively. In the sections below E denotes the reduced one-dimensional field, $E=E(z)$.

C. TM polarization

In the case of TM polarization the magnetic field $H=H(y,z)$ is perpendicular to the plane of incidence. Field H satisfies the two-dimensional nonlinear Maxwell equation

$$\frac{\partial}{\partial y}\left(\frac{1}{\epsilon}\frac{\partial H}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\epsilon}\frac{\partial H}{\partial z}\right)+H=0, \tag{4}$$

where $\epsilon(y,z)=n^2(y,z)$ is the electric permittivity of the medium. The Kerr law can now be written in the form

$$\epsilon(y,z)=\epsilon_0+\epsilon_2|E(y,z)|^2, \tag{5}$$

where $\epsilon_0=n_0^2$, $\epsilon_2=2n_0n_2$, and the intensity is determined from the formula

$$|E(y,z)|^2=\left|\frac{1}{\epsilon}\frac{\partial H}{\partial y}\right|^2+\left|\frac{1}{\epsilon}\frac{\partial H}{\partial z}\right|^2. \tag{6}$$

When the magnetic field is taken as a product of two functions,

$$H(y,z)=H(z)e^{ik_y y}, \tag{7}$$

Eq. (4) reduces to

$$\left(\frac{H'}{\epsilon}\right)'=\left(\frac{k_y^2}{\epsilon}-1\right)H, \tag{8}$$

and the intensity in Eq. (6) takes on the form

$$|E(z)|^2 = k_y^2 \left|\frac{H}{\epsilon}\right|^2 + \left|\frac{H'}{\epsilon}\right|^2. \tag{9}$$

The prime indicates differentiation with respect to z. We call Eqs. (5) and (8), together with intensity (9), the TM-polarized Kerr-Maxwell equation. This is the second nonlinear equation to which we apply the canonical perturbation theory. Throughout in the sections that follow, H denotes the reduced field that depends on z only.

III. CANONICAL THEORY

We calculate the nonlinear electromagnetic field configuration perturbatively using the canonical theory [14,15]. This is manifestly a new approach to apply classical mechanics in optics. Hamilton's theory has been employed earlier in quite different contexts, for example, in electron optics [17] and in variational calculus [18]. In our approach, we make use of the Hamilton-Jacobi theory and solve the linear optical system exactly in canonical variables that are constants of integration. When the nonlinearity is sufficiently weak, it may be treated as a small perturbation that disturbs the linear system. In most materials this is a physically reasonable assumption. The canonical perturbation theory is used to compute the corrections to the constant invariants. Since the canonical theory is simpler and more directly illustrated with TE polarization, we discuss this situation first. The application to TM fields is examined in Sec. III A.

A. TE polarization

We commence by showing the relation between classical mechanics and optics. We introduce a Lagrangian that describes the present one-dimensional optical system and perform a Legendre transformation leading to Hamilton's theory. We determine the exact solution of the linear system in terms of constants of integration, and subsequently apply the canonical perturbation theory to the nonlinear interaction.

As is known from the calculus of variations, optical systems can be described within the Lagrangian formalism. We consider the following Lagrangian

$$L = \left|\frac{\partial E}{\partial z}\right|^2 - c_1|E|^2 - \frac{c_2}{2}|E|^4, \tag{10}$$

where $E=E(z)$ is the complex-valued field, and c_1 and c_2 are constant coefficients. The Euler-Lagrange equation for E^* then reproduces the nonlinear TE polarized Kerr-Maxwell equation, Eq. (3). Hence the Lagrangian (10) includes all the physics needed to describe a nonlinear TE polarized system.

Our interest is to find a Hamiltonian that corresponds to Eq. (10). Thus we express the complex electric field with two real functions $\varepsilon(z)$ and $\varphi(z)$, i.e., $E(z) = \varepsilon(z)\exp[i\varphi(z)]$. The Lagrangian of Eq. (10) transforms into

$$L = L(\varepsilon, \varepsilon', \varphi') = \varepsilon'^2 + \varphi'^2\varepsilon^2 - c_1\varepsilon^2 - \frac{c_2}{2}\varepsilon^4, \tag{11}$$

where, as before, the prime denotes differentiation with respect to z. As in conventional classical mechanics, we may define two pairs of canonical variables

$$(\varepsilon, p) \quad \text{and} \quad (\varphi, p_c), \tag{12}$$

where the conjugate momenta are derived from the Lagrangian as

$$p = \frac{\partial L}{\partial \varepsilon'} = 2\varepsilon', \quad p_c = \frac{\partial L}{\partial \varphi'} = 2\varphi'\varepsilon^2. \tag{13}$$

The Hamiltonian is now found by applying the Legendre transformation to Lagrangian (11),

$$H_{\text{total}}(\varepsilon, p, p_c) \equiv \varepsilon' p + \varphi' p_c - L = \frac{p^2}{4} + \frac{p_c^2}{4\varepsilon^2} + c_1\varepsilon^2 + \frac{c_2}{2}\varepsilon^4. \tag{14}$$

The right-hand side resembles classical systems with a quadratic momentum p^2 and with potentials ε^2 and ε^4. However, the second term $(p_c/\varepsilon)^2$ is rather specific because it does not have a counterpart among the conventional systems of classical mechanics. In optics, the origin of this term can be traced to the phase factor of complex field E.

The total Hamiltonian can be divided into two parts: $H_{\text{total}} = H_0 + H_1$. Here H_0 is the Hamiltonian of the linear system $(c_2 = 0)$

$$H_0 = \frac{p^2}{4} + \frac{p_c^2}{4\varepsilon^2} + c_1\varepsilon^2 = E_0, \tag{15}$$

and the influence of the nonlinearity is described by

$$H_1 = \frac{c_2}{2}\varepsilon^4. \tag{16}$$

Constant E_0 represents the ''energy'' of the linear system. It is assumed that the solution corresponding to Eq. (15) does not vanish at the boundaries of a layered structure, from which it follows that E_0 must be (positive) nonzero.

Let us next determine the analytical form of the electric field of the linear system H_0. The customary solution consists of two counterpropagating plane waves. In our approach we write the same analytical solution using canonical variables that are constants of integration, i.e., independent of z. This specific form of the solution makes it possible to apply the canonical perturbation theory to the nonlinear part H_1 (perturbation). When the nonlinearity is weak, the changes to the linear theory are expected to be small. The theoretical background of the canonical theory is based on the Hamilton-Jacobi equation, which describes the canonical transformation to constant variables [14,15]. When solving the Hamilton-Jacobi equation, we simultaneously get an exact analytical solution to the linear problem.

What are the constants of integration? The Lagrangian of Eq. (11) (with $c_2 = 0$) does not include the generalized coordinate φ and so the corresponding momentum p_c already is a conserved quantity. If the Euler-Lagrange equation is formed for φ, the result is

$$\frac{d}{dz}\left(\frac{\partial L}{\partial \varphi'}\right)=\frac{d}{dz}p_c=0. \tag{17}$$

This shows that p_c indeed is a constant with respect to z.

We now use the Hamilton-Jacobi theory to transform the canonical pair (p,ε) to variables that are constants (α,β). With the help of Eq. (15) we form the Hamilton-Jacobi equation for S (Hamilton's principal function) [14],

$$\frac{1}{4}\left(\frac{\partial S}{\partial \varepsilon}\right)^2+\frac{p_c^2}{4\varepsilon^2}+c_1\varepsilon^2+\frac{\partial S}{\partial z}=0. \tag{18}$$

The solution to Eq. (18) is found in the form

$$S(\varepsilon,\alpha,z)=W(\varepsilon,\alpha)-\alpha z, \tag{19}$$

where $W(\varepsilon,\alpha)$ is Hamilton's characteristic function and α is a constant. Equation (19) is substituted into Eq. (18), resulting in

$$\frac{1}{4}\left(\frac{\partial W}{\partial \varepsilon}\right)^2+\frac{p_c^2}{4\varepsilon^2}+c_1\varepsilon^2=\alpha. \tag{20}$$

Constant α may thus be identified with energy E_0, i.e.,

$$H_0=\alpha. \tag{21}$$

Since H_0 is not an explicit function of z, energy E_0 (and thus α) is one constant of integration. On solving Eq. (20) with respect to W,

$$W=2\sqrt{\alpha}\int d\varepsilon\sqrt{1-\frac{p_c^2/(4\varepsilon^2)+c_1\varepsilon^2}{\alpha}}, \tag{22}$$

we find that S in Eq. (19) becomes

$$S=2\sqrt{\alpha}\int d\varepsilon\sqrt{1-\frac{p_c^2/(4\varepsilon^2)+c_1\varepsilon^2}{\alpha}}-\alpha z. \tag{23}$$

One could perform the integration, but that is not required to solve the Hamilton-Jacobi equation.

The third constant of integration, β, is obtained by differentiating Hamilton's principal function with respect to α, i.e.,

$$\beta=\frac{\partial S}{\partial \alpha}=\frac{1}{\sqrt{\alpha}}\int d\varepsilon\frac{1}{\sqrt{1-(p_c^2/(4\varepsilon^2)+c_1\varepsilon^2)/\alpha}}-z. \tag{24}$$

Carrying out the integration with respect to ε yields

$$z+\beta=\frac{1}{2\sqrt{c_1}}\arcsin\left[\frac{2c_1\varepsilon^2-\alpha}{\sqrt{\alpha^2-p_c^2c_1}}\right]. \tag{25}$$

On inverting Eq. (25) we find an expression for the envelope of the electric field

$$\varepsilon=\left[\frac{1}{2c_1}\{\alpha+\sqrt{\alpha^2-p_c^2c_1}\sin[2\sqrt{c_1}(z+\beta)]\}\right]^{1/2}. \tag{26}$$

The complete electric-field solution includes also the phase factor φ. From Eqs. (13) and (26) we obtain

$$\varphi=\frac{p_c}{2}\int\frac{1}{\varepsilon^2}dz=\tan^{-1}\left[\frac{\alpha}{p_c\sqrt{c_1}}\tan[\sqrt{c_1}(z+\beta)]+\frac{1}{p_c}\sqrt{\frac{\alpha^2-p_c^2c_1}{c_1}}\right]+\varphi_0, \tag{27}$$

where φ_0 is a constant. We note that $\varphi-\varphi_0$ may be positive or negative depending on the sign of p_c (energy α is always positive).

The field momentum p, which is canonical pair to ε, is obtained in a similar way from Eqs. (19), (20), and (26). The result is

$$p=\frac{\partial S}{\partial \varepsilon}=\frac{\partial W}{\partial \varepsilon}=2\sqrt{\alpha-\frac{p_c^2}{4\varepsilon^2}-c_1\varepsilon^2},$$

$$=\sqrt{2}\sqrt{\frac{(\alpha^2-c_1p_c^2)\cos^2[2\sqrt{c_1}(z+\beta)]}{\alpha+\sqrt{\alpha^2-c_1p_c^2}\sin[2\sqrt{c_1}(z+\beta)]}}. \tag{28}$$

One may verify that envelope (26) and momentum (28) indeed satisfy the Hamiltonian of the linear system, Eq. (15). Hence we conclude that by using the Jacobi-Hamilton theory we have found an exact solution for the electric field ($c_2=0$) in terms of constants of integration α, β, and p_c.

Once the rigorous solution is known, we may determine all unknown quantities of the linear system in terms of constants of integration. For example, the Lagrangian of Eq. (11) assumes the form

$$L=-\sqrt{\alpha^2-c_1p_c^2}\sin[2\sqrt{c_1}(z+\beta)], \tag{29}$$

and Hamilton's principal function S becomes

$$S(\alpha,\beta,p_c)=\int L\,dz+\text{const}$$

$$=\frac{1}{2\sqrt{c_1}}\sqrt{\alpha^2-c_1p_c^2}\cos[2\sqrt{c_1}(z+\beta)]+\text{const.} \tag{30}$$

In both equations we have used the condition $c_2=0$.

We are now in a position to apply the canonical perturbation theory to the nonlinear part H_1. The canonical property of a given coordinate transformation is independent of the particular form of the Hamiltonian. Therefore the transformation $(p,\varepsilon)\rightarrow(\alpha,\beta)$ generated by $S(\varepsilon,\alpha,z)$ remains a canonical transformation for the perturbed system [14]. Only now the new Hamiltonian H_{total} will no longer be constant with respect to α and β. The equations of motion for the transformed variables are

$$\alpha'=-\frac{\partial H_1}{\partial\beta}\quad\text{and}\quad\beta'=\frac{\partial H_1}{\partial\alpha}. \tag{31}$$

When H_1 is taken according to its definition, Eqs. (16) and (26), one obtains

$$\alpha'=-\frac{c_2}{\sqrt{c_1}}\sqrt{\alpha^2-c_1p_c^2}\cos[2\sqrt{c_1}(z+\beta)]\varepsilon^2 \tag{32}$$

and

$$\beta'=\frac{c_2}{2c_1}\left(1+\frac{\alpha\sin[2\sqrt{c_1}(z+\beta)]}{\sqrt{\alpha^2-c_1p_c^2}}\right)\varepsilon^2. \tag{33}$$

Generally, both "constants" α and β will depend on the field intensity $|E|^2=\varepsilon^2$ in the perturbed system. However, when the nonlinearity is weak, as is normally the case in optics, only the first corrections to α_0 and β_0 are important, where the subscripts denote the constant unperturbed (linear) values of α and β.

Hence, as a final result, we determine the first-order corrections to α and β. They are immediately obtained from expressions (32) and (33) via a direct substitution of the linear values α_0 and β_0 on the right-hand sides, i.e., $\alpha_1'=\alpha'|_0$ and $\beta_1'=\beta'|_0$. However, these forms of the solutions are not particularly convenient because of the rapid oscillatory motions. It is customary to calculate a net value. For a d-periodic function $g(z)=g(z+d)$, the average value is defined as

$$\overline{g}=\frac{1}{d}\int_0^d g(z)\,dz. \tag{34}$$

From Eqs. (32) and (33) we may identify the period to be $d=\pi/\sqrt{c_1}$. Hence the secular perturbation at a constant rate is given by

$$\overline{\alpha_1'}=0\quad\text{and}\quad\overline{\beta_1'}=c_2\frac{3\alpha_0}{8c_1^2}. \tag{35}$$

These equations are integrated over z resulting in

$$\overline{\alpha_1}=\alpha_0\quad\text{and}\quad\overline{\beta_1}=\beta_0+c_2\frac{3\alpha_0}{8c_1^2}z. \tag{36}$$

We observe that the Kerr-type nonlinearity does not change the "energy" of the system (α_0) to first order.

According to our perturbative method the form of the electric field [Eqs. (26) and (27)] remains unchanged but the constants of integration are modified by the interaction. On using Eq. (36) the electric field $E(z)=\varepsilon(z)\exp[i\varphi(z)]$, correct to first order by perturbation due to a (weak) nonlinearity, thus is

$$\varepsilon(z)=\left\{\frac{1}{2c_1}\left[\alpha_0+\sqrt{\alpha_0^2-p_c^2c_1}\times\sin\left\{2\sqrt{c_1}\left[z+\left(\beta_0+c_2\frac{3\alpha_0}{8c_1^2}z\right)\right]\right\}\right]\right\}^{1/2}, \tag{37a}$$

$$\varphi(z)=\tan^{-1}\left(\frac{\alpha_0}{p_c\sqrt{c_1}}\tan\left\{\sqrt{c_1}\left[z+\left(\beta_0+c_2\frac{3\alpha_0}{8c_1^2}z\right)\right]\right\}+\frac{1}{p_c}\sqrt{\frac{\alpha_0^2-p_c^2c_1}{c_1}}\right)+\varphi_0, \tag{37b}$$

where the four constants α_0, β_0, p_c, and φ_0 are determined from boundary conditions, as usual. Equations (37a) and (37b) are one of the main results of this paper. The nonlinearity alters the period of the oscillatory motion but leaves the strength of the intensity unchanged. The corresponding linear result is recovered in the limit $c_2\rightarrow 0$. When the nonlinearity is increased it is reasonable to assume that the interactions couple also to the "field energy" and α_0 is modified. In this situation one could integrate Eqs. (32) and (33) for an exact result or calculate more corrections for the expansion by perturbation. The latter is accomplished by substituting the values of Eq. (36) onto the right-hand sides of Eqs. (32) and (33) and repeating the procedure above.

The analytical results, Eqs. (37), are numerically verified in Sec. V using the exact nonlinear thin-layer theory developed for planar structures with Kerr nonlinearities. The effects of changes in the parameters of the nonlinear medium are also assessed.

B. TM polarization

The use of canonical perturbation theory with TE-polarized fields is rather straightforward, mainly because the Lagrangian (10) has a simple form. In the case of TM polarization, the corresponding Lagrangian is more involved and one is advised to consider a different approach. We use the following idea: we express the exact quadrature solution of the TM-polarized Kerr-Maxwell equation in terms of the electric field and find its relation to the theory of TE-polarized waves. Both theories, TE and TM, are mathematically equivalent at normal incidence. On expanding the fields in Taylor series at a small angle one gets the nonlinear interactions that characterize the electric field of TM-polarized waves, and a perturbative solution up to first order is found.

In this technique, some additional assumptions are made; hence the final result is expected to be slightly less accurate than that in TE polarization.

A general form of the exact quadrature solution of the TM-polarized nonlinear Maxwell equation is given in Appendix A [Eq. (A13)]. When the nonlinear layer is taken to be a Kerr medium, the solution for the magnetic-field envelope becomes [from Eqs. (A2), (A13a), (A14), and (A15)],

$$h^2=\frac{\epsilon}{2k_y^2-\epsilon}[\epsilon I-J+C_2], \tag{38}$$

where C_2 is constant and

$$h^2=|H|^2, \tag{39}$$

$$\epsilon=\epsilon_0+\epsilon_2|E|^2, \tag{40}$$

$$I=\frac{\epsilon-\epsilon_0}{\epsilon_2}=|E|^2, \tag{41}$$

$$J=\frac{(\epsilon-\epsilon_0)^2}{2\epsilon_2}=\frac{\epsilon_2|E|^4}{2}. \tag{42}$$

Using the notations above, solution (38) can be rewritten as

$$|H|^2=\frac{\epsilon_0+\epsilon_2|E|^2}{2k_y^2-(\epsilon_0+\epsilon_2|E|^2)}\left[(\epsilon_0+\epsilon_2|E|^2)|E|^2-\frac{\epsilon_2|E|^4}{2}+C_2\right]. \tag{43}$$

The only term that involves field H is on the left-hand side of Eq. (43). We would like to replace it by some combination of field E and its derivative. By using the TM-polarized Maxwell equations and Eq. (7) we obtain

$$|E'|^2=\left|\left(\frac{k_y^2}{\epsilon}-1\right)H\right|^2+k_y^2\left|\left(\frac{H}{\epsilon}\right)'\right|^2, \tag{44}$$

where $|E'|^2$ is defined in accordance with Eq. (6). We observe that if $k_y^2=0$, Eqs. (43) and (44) reduce to

$$|E'|^2+\epsilon_0|E|^2+\frac{\epsilon_2}{2}|E|^4=-C_2. \tag{45}$$

This is exactly the TE-polarized total Hamiltonian, Eq. (14), when we identify $c_1=\epsilon_0$, $c_2=\epsilon_2$, and energy $E_0=-C_2$.

Let us next consider the boundary conditions. When the wave is TM polarized, the boundary conditions for a one-dimensional field $H(z)$ are

$$H_1|_{z_0}=H_2|_{z_0}, \tag{46}$$

$$\frac{1}{\epsilon_1}\frac{\partial H_1}{\partial z}\bigg|_{z_0}=\frac{1}{\epsilon_2}\frac{\partial H_2}{\partial z}\bigg|_{z_0}, \tag{47}$$

where the subscripts denote media 1 and 2, and z_0 represents the boundary. If the incident wave is perpendicular to the structure $(k_y=0)$, one can show from Maxwell's equations that $E=E_y=i\epsilon^{-1}\partial H/\partial z$. Thus the boundary conditions (46) and (47) transform as

$$\frac{\partial E_1}{\partial z}\bigg|_{z_0}=\frac{\partial E_2}{\partial z}\bigg|_{z_0}, \tag{48}$$

$$E_1|_{z_0}=E_2|_{z_0}. \tag{49}$$

In the former equality we also used the nonlinear Maxwell equation (4). Equations (48) and (49) are recognized as the usual boundary conditions for TE-polarized waves. Thus we have shown that Eq. (45), together with Eqs. (48) and (49), relates the TE and TM polarizations; the mathematical formalism is the same for both polarizations when the field is normally incident.

We investigate a more general situation in which the incident wave propagates at an angle to the normal of the structure $(k_y\neq 0)$. The mutual interactions of the fields make the general solution of Eq. (43) complicated. We assume that k_y is a small parameter and Eq. (44) can be taken as

$$|E'|^2\approx|H|^2. \tag{50}$$

This is an additional approximation to keep the problem soluble. From a physical point of view, the assumption neglects the angular dependence in the kinetic part of the Lagrangian, i.e., nonzero k_y modifies only the nonlinear potential. The boundary conditions, Eqs. (48) and (49), are also taken to remain approximatively valid for small k_y. In the numerical example in Sec. V we will justify the validity of these assumptions.

Using Eq. (50) and a small k_y condition we systematically expand Eq. (43) in a Taylor series with respect to ϵ_2. To first order the result is

$$|E'|^2+\frac{\epsilon_0^2}{\epsilon_0-2k_y^2}|E|^2+\frac{\epsilon_2(\epsilon_0^2-6\epsilon_0k_y^2)}{2(\epsilon_0-2k_y^2)^2}|E|^4+\frac{C_2}{\epsilon_0-2k_y^2}\left[\epsilon_0-\frac{\epsilon_2 2k_y^2}{\epsilon_0-2k_y^2}|E|^2\right]=0. \tag{51}$$

All fields are in quadratic forms, so we may directly use the canonical perturbation theory developed in Sec. III A.

We set $\epsilon_2=0$ in Eq. (51) and introduce an unperturbed Hamiltonian

$$H_0\equiv\frac{p^2}{4}+\frac{p_c^2}{4\varepsilon^2}+\tilde{c}_1\varepsilon^2=\tilde{E}_0, \tag{52}$$

where the constant coefficients are

$$\tilde{c}_1=\frac{\epsilon_0^2}{\epsilon_0-2k_y^2}\quad\text{and}\quad\tilde{E}_0=-\frac{C_2\epsilon_0}{\epsilon_0-2k_y^2}. \tag{53}$$

The inhomogenous (perturbation) part consists of the terms that are proportional to the nonlinearity coefficient ϵ_2,

$$H_1 \equiv \epsilon_2 \left[-\frac{2C_2k_y^2}{(\epsilon_0-2k_y^2)^2}\varepsilon^2 + \frac{\epsilon_0^2-6\epsilon_0k_y^2}{2(\epsilon_0-2k_y^2)^2}\varepsilon^4 \right]. \quad (54)$$

As before, we write the perturbation Hamiltonian H_1 with canonical variables α, β, and p_c,

$$H_1 = -\frac{\epsilon_2 2C_2k_y^2}{(\epsilon_0-2k_y^2)^2}\left[\frac{1}{2\tilde{c}_1}\{\alpha+\sqrt{\alpha^2-p_c^2\tilde{c}_1} \times \sin[2\sqrt{\tilde{c}_1}(z+\beta)]\}\right] + \frac{\epsilon_2(\epsilon_0^2-6\epsilon_0k_y^2)}{2(\epsilon_0-2k_y^2)^2} \times \left[\frac{1}{2\tilde{c}_1}\{\alpha+\sqrt{\alpha^2-p_c^2\tilde{c}_1}\sin[2\sqrt{\tilde{c}_1}(z+\beta)]\}\right]^2. \quad (55)$$

The corrections are calculated similarly as in the case of TE polarization. One can show that the averaged changes of α' and β', to first order, become

$$\overline{\alpha_1'} = 0 \quad \text{and} \quad \overline{\beta_1'} = \epsilon_2 \frac{2\tilde{c}_1C_3+3\alpha_0C_4}{4\tilde{c}_1^2}, \quad (56)$$

where C_3 and C_4 are constant coefficients defined as

$$C_3 = -\frac{2C_2k_y^2}{(\epsilon_0-2k_y^2)^2} \quad \text{and} \quad C_4 = \frac{\epsilon_0^2-6\epsilon_0k_y^2}{2(\epsilon_0-2k_y^2)^2}. \quad (57)$$

Again we have neglected the higher-order perturbation terms. This is possible due to the assumption that H_1 be small. We illustrate the variation of α_1' and its averaged value $\overline{\alpha_1'}$, from Eq. (56), in Fig. 2(a). Similarly, β_1' and $\overline{\beta_1'}$ are shown in Fig. 2(b) over three cycles, $z=3d$. In both figures we have used, for simplicity, the values $c_1=2$, $C_3=1$, $C_4=1$, and $\sqrt{\alpha_0^2-c_1p_c^2}=1$.

Expressions in Eq. (56) are readily integrated, resulting in

$$\overline{\alpha_1} = \alpha_0 \quad \text{and} \quad \overline{\beta_1} = \beta_0 + \epsilon_2 \frac{2\tilde{c}_1C_3+3\alpha_0C_4}{4\tilde{c}_1^2}z, \quad (58)$$

and these values are to be substituted into the unperturbed formulas of the electric-field amplitude and phase, Eqs. (26) and (27). This is our main result for TM polarization. The first-order corrections to the propagation invariants and the resulting field are comparable to the corresponding formulas for TE polarization, Eqs. (36) and (37). If the nonlinearity is increased sufficiently, the assumption of small H_1 may break down. In this situation the perturbation Hamiltonian can be expanded further in a Taylor series and the corrections calculated to higher orders.

We note that there is also another way to define the complete perturbed Hamiltonian, H_{total}. The inhomogenous part H_1 is chosen to consist of the second term of Eq. (54) alone while the first term is absorbed into coefficient $\tilde{c}_1$. This choice mixes the linear and nonlinear modes in the homogenous Hamiltonian H_0. The effect of the perturbation Hamiltonian H_1 becomes weaker. However, we have chosen to

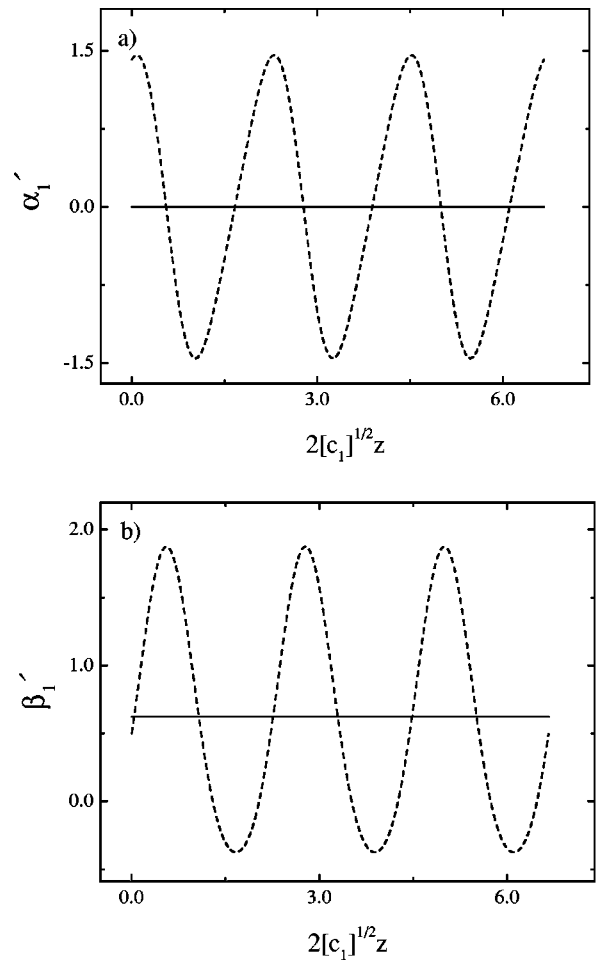

FIG. 2. First-order corrections of the propagation invariants; (a) α_1' and (b) β_1'. The rigorous values are represented by dashed lines and the corresponding averaged results are illustrated with solid lines. The ensemble is taken over three periods and the fields are TM polarized.

separate all nonlinear terms from H_0 in order to keep the mathematical formalism clear.

In Sec. V we give a numerical example to demonstrate the validity of the analytical perturbative formulas, Eq. (58). We also compare these TM results to the corresponding values of TE polarization.

IV. THIN-LAYER THEORY

The accuracy of the approximative analytical solutions, Eqs. (36), (37), and (58), is readily assessed by the exact nonlinear thin-layer theory. This is a new method that was presented, mainly for TE-polarized fields, in Ref. [16]. In this section we briefly summarize the main features of the method, with an emphasis on TM polarization.

A. TE polarization

The general idea of the thin-layer theory is that the nonlinear medium is divided into N thin slabs [16]. In each individual slab j $(j=1,\ldots,N)$, the nonlinear Kerr-Maxwell equation (3) is exactly solved using a trial function

$$E_j(z)=A_j\cos(k_{j,1}z)+\frac{iB_j}{k_{j,2}}\sin(k_{j,2}z) \tag{59}$$

and a thin-layer approximation $(k_{j,l}\Delta z)^2\approx 0 \ \ (l=1,2)$. Here A_j and B_j are constant amplitudes, and $k_{j,1}$ and $k_{j,2}$ are propagation constants.

The boundary conditions between two consecutive thin slabs determine the characteristic matrix of the nonlinear medium

$$\begin{bmatrix} A_j \\ B_j \end{bmatrix}=\begin{bmatrix} 1 & ih \\ ik_{j+1,1}^2 h & 1 \end{bmatrix}\begin{bmatrix} A_{j+1} \\ B_{j+1} \end{bmatrix}, \tag{60}$$

where h is the slab thickness. Wave vector $k_{j+1,1}^2=c_1+c_2|A_{j+1}|^2$ is amplitude dependent, in contrast to the conventional linear theory [19]. Since the wave vector is not constant (it depends on the value of the electric field), the numerical computation is initiated from the back surface of the nonlinear medium. In this way one can propagate the wave through the layer and find (without iteration) the electromagnetic field solution on the front boundary of the nonlinear medium. The accuracy of the technique is assessed and more detailed physical explanations are given in Ref. [16].

B. TM polarization

The waves in TM polarization are treated substantially similarly. However, a further assumption is needed, namely that the refractive index be constant in each separate thin slab, i.e., $\partial\epsilon_j/\partial z=0 \ \ (j=1,\dots,N)$. In the limit of zero layer thickness this assumption will be valid. Now the nonlinear Kerr-Maxwell equation (8), together with Eqs. (5) and (9), is solved using a trial function

$$H_j=A_j\cos(k_jz)+\frac{i\epsilon_jB_j}{k_j}\sin(k_jz) \tag{61}$$

and a thin-layer approximation $(k_j\Delta z)^2\approx 0$. Here A_j and B_j are constants, and k_j is a wave vector in the jth nonlinear thin slab.

Again, the numerical computation of the nonlinear solution is initiated from the back surface of the nonlinear layer. The transmitted amplitude is taken as some arbitrary complex number, after which amplitude coefficients A_N and B_N are calculated with the help of the boundary conditions between the linear medium and the last nonlinear slab. Using Eq. (9) and the assumption of constant ϵ_j, Eq. (5) is written in the form

$$\begin{aligned}\epsilon_j&=\epsilon_0+\epsilon_2\left(k_y^2\left|\frac{H_j}{\epsilon_j}\right|^2+\left|\frac{H_j'}{\epsilon_j}\right|^2\right),\\ &=\epsilon_0+\epsilon_2\left(k_y^2\frac{|A_j|^2}{|\epsilon_j|^2}+|B_j|^2\right),\end{aligned} \tag{62}$$

where $\epsilon_0=n_0^2$ and $\epsilon_2=2n_0n_2$. Equation (62) is a third-order polynomial for ϵ_j and its root can be found by numerical [20] or analytical (Appendix B) means.

When the root of ϵ_N is known, wave vector k_N may be computed using Eq. (8), i.e., $k_j^2=\epsilon_j-k_y^2$. We may then construct the TM-polarized characteristic matrix,

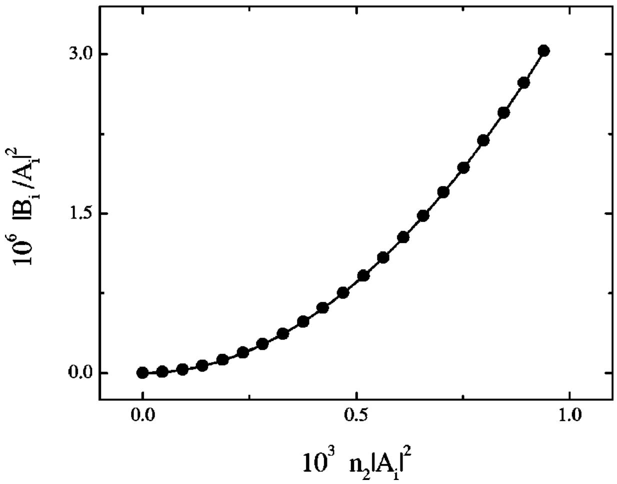

FIG. 3. Scattered intensity for the exact theory (dots) and the perturbative result (solid line), when the incident field is TE polarized, $n'=1.45$, $n_0=1.6$, and $d\approx 2.2\lambda$. The incident and the reflected intensities are denoted by $|A_i|^2$ and $|B_i|^2$, respectively.

$$\begin{pmatrix} A_j \\ B_j \end{pmatrix}=\begin{pmatrix} 1 & i\epsilon_{j+1}h \\ ik_{j+1}^2h/\epsilon_{j+1} & 1 \end{pmatrix}\begin{pmatrix} A_{j+1} \\ B_{j+1} \end{pmatrix}, \tag{63}$$

where h is the slab thickness. Amplitudes A_{N-1} and B_{N-1} are obtained using Eq. (63).

This procedure, Eqs. (62) and (63), is repeated for each nonlinear layer j. Finally the incident and the reflected amplitudes, A_i and B_i, are determined at the boundary of the front linear layer and the first nonlinear slab. The plot $|B_i/A_i|^2$ versus $n_2|A_i|^2$ uniquely illustrates the intensity reflected off the nonlinear structure as a function of the input-wave intensity.

V. NUMERICAL RESULTS

We investigate nonlinear wave reflection in a three-layer structure, Fig. 1 with $n''=n'$. These types of systems have important applications in modern optoelectronics, for example, as waveguides and nonlinear mirrors [13]. We verify the validity of the perturbative solutions of the canonical theory by comparing them to the exact nonlinear thin-layer results.

A. TE polarization

In the first numerical example the refractive indices of the media are taken $n'=1.45$ and $n_0=1.6$ (for illustrative purposes). An incident plane wave of wavelength $\lambda=1.06\times 10^{-6}$ m propagates perpendicularly to the structure $\theta=0$. The thickness of the nonlinear middle layer is chosen so that in the linear case [$n_2=0$, see Eq. (2)] all light passes the structure $d\approx 2.2\lambda$. The parameter that is varied during the computation is the nonlinear coefficient n_2. The reflected intensities in the two different nonlinear theories are illustrated in Fig. 3 as a function of $n_2|A_i|^2$, where $|A_i|^2$ is the incident-wave intensity. The solid line represents the analytical formulas of the canonical perturbation theory, Eqs. (37), while the dots are the corresponding exact numerical results of the nonlinear thin-layer theory. In the latter method, one typically needs about 10^5 thin layers per wavelength. Figure

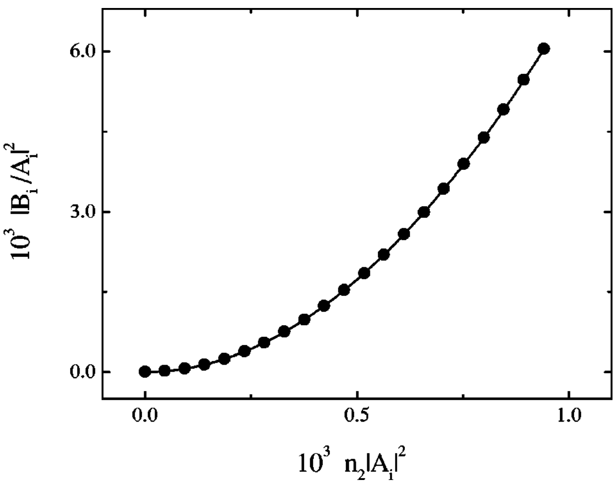

FIG. 4. Scattered intensity. The thickness of the nonlinear layer is $d \approx 31\lambda$, but otherwise the parameters and illustrations are as in Fig. 3.

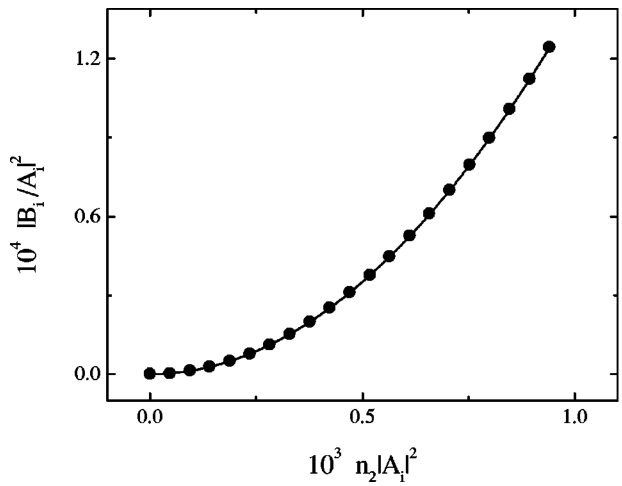

FIG. 5. Scattered intensity. The index of the middle layer is $n_0 = 3.6$ and $d \approx 2.1\lambda$. Parameter n', the dots, and the line are as in Fig. 3.

3 shows that the perturbative results coincide with the exact numerical results. We also note that $n_2|A_i|^2$ is a dimensionless quantity and that for physical systems $n_2|E|^2 \ll 1$. This condition fixes the upper limit of the horizontal scale in Fig. 3.

When the nonlinear coefficient $n_2 = 0$, there is no reflection and the solution to Maxwell's equations in the middle layer can be expressed with two counterpropagating plane waves. The corresponding optical intensity then is a periodic function of index n_0. When the nonlinearity is added, n_2 changes the period of oscillations according to Eq. (37a). The fields fall off from exact resonance and the scattered intensity increases (see Fig. 3). In our numerical example for nonzero n_2 the relative amount of the reflected light is quite small, less than 10^{-5}. The small value of scattering is entirely due to the specific system parameters used in the computation.

Next we change the nonlinear layer thickness to $d \approx 31\lambda$, but otherwise retain the same system parameters as before. The results of the reflection calculations using the two nonlinear theories are now shown in Fig. 4. The solid line and dots are the perturbative and exact results, respectively. The results of both nonlinear methods again match in details. In general, the scattered intensity increases when the nonlinear layer becomes thicker, which is due to the longer spatial regime of nonlinear interaction. The analytical explanation, which can be seen from Eq. (37a), is that the ''nonlinear intensity'' oscillates with a different period as compared to the corresponding linear intensity, and the differences become larger when the distance increases. We may further deduce that if the interaction region is sufficiently long and the refractive indices are properly chosen, the nonlinear field may oscillate an extra cycle causing a bistable effect in the scattered fields.

Finally, we change the index of refraction to $n_0 = 3.6$ and choose thickness $d \approx 2.1\lambda$ for the nonlinear layer. The corresponding scattering results are given in Fig. 5. The system parameters are comparable with those used in Fig. 3. When the index jump between the linear and nonlinear media becomes large, more light is reflected by the layered structure. But the perturbative analytical solution, Eq. (37), again gives a very accurate estimate for the field within the nonlinear medium.

We conclude that while nonlinearity influences the scattered intensities, for TE waves the first-order perturbative analytical solution to the nonlinear fields, Eq. (37), already is in an excellent agreement with the exact theory. The accuracy of the result is independent of the index and layer thickness of the nonlinear medium. Hence the analytical perturbative formulas can reliably be used for the evaluation of wave reflection and propagation in nonlinear layered devices. A significant advantage of the canonical theory over the thin-layer theory relates to the computation time. The calculation of a thin-layer problem typically takes several minutes of CPU time on a conventional PC, whereas the perturbative result is obtained immediately.

B. TM polarization

We now perform a similar scattering analysis for fields in TM polarization. The first-order analytical expressions are given by Eq. (58), together with Eqs. (26) and (27). The system parameters are taken as follows: $n' = 1.45$, $n_0 = 1.6$, $\lambda = 1.06 \times 10^{-6}$ m, $\theta = 6°$, and $d \approx 2.2\lambda$. The scattered intensities are illustrated in Fig. 6. The solid line corresponds to a perturbative solution and the dots are obtained by using the TM-polarized nonlinear thin-layer theory. As in the case of TE polarization, the reflected intensity vanishes in the linear limit and it increases when n_2 is varied.

The two nonlinear theories deviate from each other when the nonlinearity is sufficiently large. There are many reasons for this small diversion. The angle of incidence is relatively large, $\theta = 6°$, which gives $k_y^2 \approx 0.023$. So we have reached the limit of k_y^2 being small, as was assumed in the derivation of perturbative results in Sec. III B. Assumption (50) and boundary conditions (48) and (49) are slightly violated, but these approximations are still quite reasonable for small θ. In addition, the higher-order corrections in the expansion by perturbation may also alter the result. But clearly the perturbative analytical solution, Eq. (58), provides a useful, tangible expression that incorporates the nonlinear effects characterizing TM polarization. Changes in the index or layer

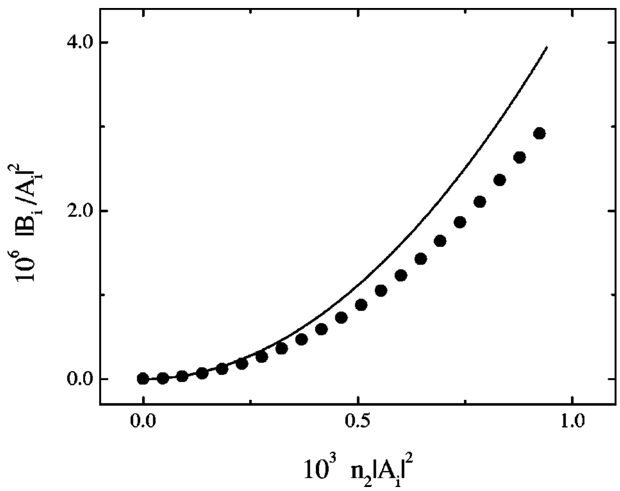

FIG. 6. Scattered intensity for the exact theory (dots) and the perturbative result (solid line), when the incident field is TM polarized, $\theta=6°$, $n_0=1.6$, and $d\approx 2.2\lambda$.

thickness of the nonlinear medium give results that are graphically very similar to those in Fig. 6.

Hence the simulation results show that the first-order perturbative analytical expression for the electric field in TM polarization, Eqs. (26), (27), and (58), is not quite as accurate as the corresponding formula for a TE-polarized wave, but still in most cases it gives a good estimate for the TM-polarized field in a nonlinear medium.

VI. CONCLUSIONS

We have investigated a three-layered structure, in which a nonlinear Kerr medium is sandwiched by two linear media. The incident light is either TE or TM polarized. Similar systems have many applications, e.g., in contemporary optoelectronic devices. Especially, we studied the nonlinear wave reflection in layered structures. We used two different methods; a perturbation theory of classical mechanics and an exact nonlinear thin-layer theory. Our analytical and numerical results show that a weak nonlinearity changes the period of oscillation of the field within a nonlinear layer. This has many consequences. For instance, by using a suitable nonlinear medium one can tune an optical system on and off resonance. If the indices of refraction and the thickness of the nonlinear medium are properly chosen, a bistable output is possible. Our analysis is not restricted to a sample of three media only, but it can readily be extended to multi-layered structures.

We used Hamilton's canonical theory to solve the nonlinear field equations perturbatively. This is manifestly a new approach to apply classical mechanics in optics. With the help of the Hamilton-Jacobi theory the electromagnetic fields of a linear system are expressed with constant canonical variables. The nonlinearity is treated as a small perturbation that modifies the canonical invariants. The ''optical Hamiltonian'' includes a term $1/\varepsilon^2$, which makes it generally interesting, because such Hamiltonians do not have a counterpart in customary systems of classical mechanics. In perturbative approaches the convergence is not always guaranteed. Hence we compared the perturbative results to an exact nonlinear thin-layer theory. We showed that already the first-order correction by perturbation gives very precise solutions for the nonlinear fields. The method is more directly applicable to fields in TE polarization, but it also gives information about the TM polarized waves. We note that the latter case involves an exact quadrature solution of the nonlinear field in TM polarization. Our perturbative approach has many advantages over other techniques; the nonlinear solutions have an analytical form, they are accurate, and readily implemented on a computer.

ACKNOWLEDGMENT

T.A.L. thanks Vilho, Yrjö, and Kalle Väisälä Foundation for financial support.

APPENDIX A: QUADRATURE SOLUTION

In this Appendix the TM-polarized Maxwell equation (8) with generalized permittivity

$$\epsilon=\epsilon_0+\epsilon_2 f(|E|^2) \tag{A1}$$

is exactly solved with quadratures. Here $f(|E|^2)$ is some real function of the field intensity. The formalism is similar to that in Ref. [21], but now the full phase dependence is also taken into account.

It is assumed that the nonlinear medium is nonabsorbing, i.e., ϵ_0 is real. The effective permittivity ϵ is then also a real function. The magnetic field is set as

$$H(z)=h(z)e^{i\phi(z)}, \tag{A2}$$

where $h(z)$ and $\phi(z)$ are real functions. On substituting Eq. (A2) into Eq. (8), two equations are obtained from the imaginary and real parts separately, viz.,

$$2\phi' h' \epsilon+\phi'' h\epsilon-\phi' h\epsilon'=0, \tag{A3a}$$

$$\left(\frac{h'}{\epsilon}\right)'-\frac{\phi'^2 h}{\epsilon}=\left(\frac{k_y^2}{\epsilon}-1\right)h. \tag{A3b}$$

Equation (A3a) is integrated with respect to z,

$$\frac{\phi' h^2}{\epsilon}=C_1=\text{const}, \tag{A4}$$

after which it is inserted into Eq. (A3b),

$$\left(\frac{h'}{\epsilon}\right)'-\frac{C_1^2\epsilon}{h^3}=\left(\frac{k_y^2}{\epsilon}-1\right)h. \tag{A5}$$

By using Eq. (A4), Eq. (9) becomes

$$|E|^2=k_y^2\frac{h^2}{\epsilon^2}+\frac{h'^2+(\phi' h)^2}{\epsilon^2},$$

$$=k_y^2\frac{h^2}{\epsilon^2}+\frac{h'^2}{\epsilon^2}+\frac{C_1^2}{h^2}. \tag{A6}$$

Let $I(\epsilon-\epsilon_0)$ be an inverse function of $\epsilon_2 f(|E|^2)$; it is assumed that this inverse exists. Then Eq. (A1) with Eq. (A6) can be written as

$$\left(\frac{h'}{\epsilon}\right)^2 = I(\epsilon-\epsilon_0) - k_y^2\left(\frac{h}{\epsilon}\right)^2 - \left(\frac{C_1}{h}\right)^2. \tag{A7}$$

Equation (A7) is differentiated with respect to z,

$$2\frac{h'}{\epsilon}\left(\frac{h'}{\epsilon}\right)' = I' - 2k_y^2\frac{h}{\epsilon^3}(\epsilon h' - \epsilon' h) + \frac{2C_1^2 h'}{h^3}, \tag{A8}$$

after which Eqs. (A5) and (A8) are combined resulting in

$$2\frac{hh'}{\epsilon}\left(\frac{k_y^2}{\epsilon} - 1\right) = I' - 2k_y^2\frac{h}{\epsilon^3}(\epsilon h' - \epsilon' h). \tag{A9}$$

When Eq. (A9) is multiplied by ϵ, it can be rewritten in a relatively simple form

$$\left[\left(\frac{2k_y^2-\epsilon}{\epsilon}\right)h^2\right]' = \epsilon I'. \tag{A10}$$

This is readily integrated over z,

$$h^2 = \frac{\epsilon}{2k_y^2-\epsilon}[\epsilon I(\epsilon-\epsilon_0) - J(\epsilon-\epsilon_0) + C_2], \tag{A11}$$

where

$$J(\epsilon-\epsilon_0) = \int \epsilon' I(\epsilon-\epsilon_0)dz, \tag{A12}$$

and C_2 is an integration constant.

Finally, by using Eq. (A7) the exact solution to the TM-polarized Maxwell equation, Eq. (8), is expressed as

$$h^2 = \frac{\epsilon}{2k_y^2-\epsilon}[\epsilon I(\epsilon-\epsilon_0) - J(\epsilon-\epsilon_0) + C_2], \tag{A13a}$$

$$h'^2 = \epsilon^2 I(\epsilon-\epsilon_0) - k_y^2 h^2 - \frac{C_1^2\epsilon^2}{h^2}. \tag{A13b}$$

The phase factor ϕ of the magnetic field is determined from Eq. (A4). In this analysis we have presumed that $\epsilon_2 \neq 0$ and $\epsilon_2 f(|E|^2)$ is invertible, but otherwise the solution is general. It is not restricted to any specific nonlinear medium; ϵ can be an arbitrary real function of the field intensity.

If we further specify the nonlinear medium to contain only Kerr-type nonlinearities, then

$$\epsilon = \epsilon_0 + \epsilon_2|E|^2 \tag{A14}$$

and

$$I = \frac{\epsilon-\epsilon_0}{\epsilon_2}, \quad J = \frac{(\epsilon-\epsilon_0)^2}{2\epsilon_2}, \tag{A15}$$

where ϵ_2 is related to the Kerr coefficient. We have not introduced the domain of the nonlinear equations. Hence the solution given by Eq. (A13) has two constants C_1 and C_2 that are determined by the boundary conditions.

APPENDIX B: THIRD-ORDER ROOT

When the nonlinear medium is nonabsorbing or absorption is weak, the index of refraction is uniquely determined for TM polarization, i.e., the third-order polynomial in Eq. (62) has only one real root that has a physical meaning. The general form of Eq. (62) is

$$\epsilon^3 - D_2\epsilon^2 + D_1\epsilon - D_0 = 0, \tag{B1}$$

where D_1, D_2, and D_3 are real and positive coefficients. Following the notations of Ref. [22], the quantity

$$q^3 + r^2 = \frac{1}{108}(27D_0^2 + 4D_1^3 - 18D_0D_1D_2 - D_1^2D_2^2 + 4D_0D_2^3), \tag{B2}$$

with definitions

$$q = \frac{1}{3}D_1 - \frac{1}{9}D_2^2, \tag{B3}$$

$$r = \frac{1}{6}(3D_0 - D_1D_2) + \frac{1}{27}D_2^3, \tag{B4}$$

determines the number of real roots of Eq. (B1). If $D_1 = 0$ (no absorption), expression (B2) is always positive and Eq. (B1) has one real root and a pair of complex conjugate roots. If D_1 is a small positive number (weak absorption), the root remains unique. The algebraic expression of this positive root is

$$\begin{aligned}\epsilon = \frac{1}{6}\{&2D_2 + 2^{2/3}[(27D_0 - 9D_1D_2 + 2D_2^3 \\ &- 3\sqrt{3}\sqrt{27D_0^2 + 4D_1^3 - 18D_0D_1D_2 - D_1^2D_2^2 + 4D_0D_2^3})^{1/3} \\ &+ (27D_0 - 9D_1D_2 + 2D_2^3 + 3\sqrt{3} \\ &\times \sqrt{27D_0^2 + 4D_1^3 - 18D_0D_1D_2 - D_1^2D_2^2 + 4D_0D_2^3})^{1/3}]\}.\end{aligned} \tag{B5}$$

The other two roots of Eq. (62) have imaginary parts and hence they do not have a physical significance.

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